

# Polynomial Profits in Renewable Resources Management

Rinaldo M. Colombo<sup>1</sup>

Mauro Garavello<sup>2</sup>

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## Abstract

A system of renewal equations on a graph provides a framework to describe the exploitation of a biological resource. In this context, we formulate an optimal control problem, prove the existence of an optimal control and ensure that the target cost function is polynomial in the control. In specific situations, further information about the form of this dependence is obtained. As a consequence, in some cases the optimal control is proved to be necessarily bang–bang, in other cases the computations necessary to find the optimal control are significantly reduced.

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## 1 Introduction

A biological resource is grown to provide an economical profit. Up to a fixed age  $\bar{a}$ , this population consists of *juveniles* whose density  $J(t, a)$  at time  $t$  and age  $a$  satisfies the usual renewal equation [12, Chapter 3]

$$\partial_t J + \partial_a (g_J(t, a) J) = d_J(t, a) J \quad a \in [0, \bar{a}],$$

$g_J$  and  $d_J$  being, respectively, the usual growth and mortality functions, see also [5, 6, 11]. For further structured population models, we refer for instance to [3, 4, 8, 9, 13].

At age  $\bar{a}$ , each individual of the  $J$  population is selected and directed either to the market to be sold or to provide new juveniles through reproduction. Correspondingly, we are thus lead to consider the  $S$  and the  $R$  populations whose evolution is described by the renewal equations

$$\begin{aligned} \partial_t S + \partial_a (g_S(t, a) S) &= d_S(t, a) S \\ \partial_t R + \partial_a (g_R(t, a) R) &= d_R(t, a) R \end{aligned} \quad a \geq \bar{a},$$

with obvious meaning for the functions  $g_S, g_R, d_S, d_R$ . Here, the selection procedure is described by a parameter  $\eta$ , varying in  $[0, 1]$ , which quantifies the percentage of the  $J$  population directed to the market, so that

$$\begin{aligned} g_S(t, \bar{a}) S(t, \bar{a}) &= \eta g_J(t, \bar{a}) J(t, \bar{a}) \\ g_R(t, \bar{a}) R(t, \bar{a}) &= (1 - \eta) g_J(t, \bar{a}) J(t, \bar{a}). \end{aligned}$$

<sup>1</sup>INDAM Unit, University of Brescia

<sup>2</sup>Department of Mathematics and Applications, University of Milano Bicocca

The overall dynamics is completed by the description of reproduction, which we obtain here through the usual age dependent fertility function  $w = w(a)$  using the following nonlocal boundary condition

$$g_J(t, 0) J(t, 0) = \int_{\bar{a}}^{+\infty} w(\alpha) R(t, \alpha) d\alpha .$$

In this connection, we recall the related results [1, 2, 7] in structured populations that take into consideration a juvenile–adult dynamics.

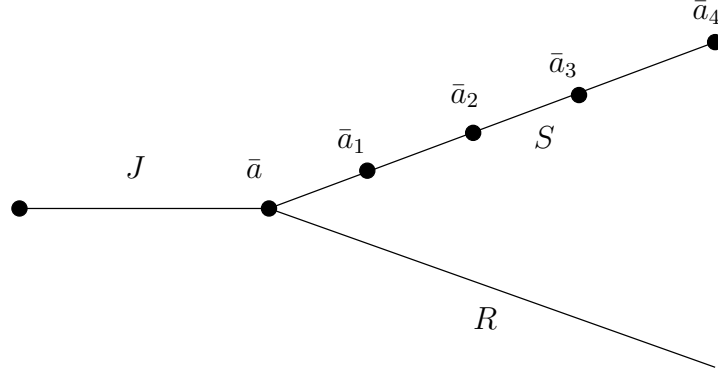


Figure 1: The graph corresponding to the biological resource. At age  $\bar{a}$ , juveniles reach the adult stage and are selected. The part  $R$  is used for reproduction. Portions of the  $S$  population are sold at ages  $\bar{a}_1, \dots, \bar{a}_4$ .

Once the biological evolution is defined, we introduce the income and cost functionals as follows. The income is related to the withdrawal of portions of the  $S$  population at given stages of its development. More precisely, we assume there are fixed ages  $\bar{a}_1, \dots, \bar{a}_N$ , with  $\bar{a} < \bar{a}_1 < \bar{a}_2 < \dots < \bar{a}_N$ , where the fractions  $\vartheta_1, \dots, \vartheta_N$  of the  $S$  population are kept, while the portions  $(1 - \vartheta_1), \dots, (1 - \vartheta_N)$  are sold. A very natural choice is to set  $\vartheta_N \equiv 0$ , meaning that nothing is left unsold after age  $\bar{a}_N$ . The dynamics of the whole system has then to be completed introducing the selection

$$S(t, \bar{a}_i +) = \vartheta_i S(t, \bar{a}_i -)$$

that takes place at the age  $\bar{a}_i$ , for  $i = 1, \dots, N$ .

Summarizing, the dynamics of the structured  $(J, S, R)$  population is thus described by the following nonlocal system of balance laws, see also Figure 1:

$$\left\{ \begin{array}{ll} \partial_t J + \partial_a (g_J(t, a) J) = d_J(t, a) J & (t, a) \in \mathbb{R}^+ \times [0, \bar{a}] \\ \partial_t S + \partial_a (g_S(t, a) S) = d_S(t, a) S & (t, a) \in \mathbb{R}^+ \times ([\bar{a}, +\infty[ \setminus \{\bar{a}_1, \dots, \bar{a}_N\}) \\ \partial_t R + \partial_a (g_R(t, a) R) = d_R(t, a) R & (t, a) \in \mathbb{R}^+ \times [\bar{a}, +\infty[ \\ g_S(t, \bar{a}) S(t, \bar{a}) = \eta g_J(t, \bar{a}) J(t, \bar{a}) & t \in \mathbb{R}^+ \\ g_R(t, \bar{a}) R(t, \bar{a}) = (1 - \eta) g_J(t, \bar{a}) J(t, \bar{a}) & t \in \mathbb{R}^+ \\ g_J(t, 0) J(t, 0) = \int_{\bar{a}}^{+\infty} w(\alpha) R(t, \alpha) d\alpha & t \in \mathbb{R}^+ \\ S(t, \bar{a}_i +) = \vartheta_i S(t, \bar{a}_i -) & t \in \mathbb{R}^+, \quad i = 1, \dots, N \\ J(0, a) = J_o(a) & a \in [0, \bar{a}] \\ S(0, a) = S_o(a) & a \in [\bar{a}, +\infty[ \\ R(0, a) = R_o(a) & a \in [\bar{a}, +\infty[ \end{array} \right. \quad (1.1)$$

where we inserted the initial data  $(J_o, S_o, R_o)$ .

Our key result is the proof that for all  $t$  and all  $a$ , the quantities  $J(t, a)$ ,  $S(t, a)$  and  $R(t, a)$  are *polynomial* in the values attained by the control parameters  $\eta$  and  $\vartheta$ .

We now pass to the introduction of the expressions of cost and income. To this aim, we first fix a time horizon  $T$ , with  $T > 0$ . Then, a reasonable expression for the income is

$$\mathcal{I}(\eta, \vartheta; T) = \int_0^{\bar{a}} P(a, J(T, a)) da + \sum_{i=1}^N \int_0^T P_i(t, (1 - \vartheta_i(t)) S(t, \bar{a}_i -)) dt. \quad (1.2)$$

The latter term above is the sum of the incomes due to the selling of the  $S$  individuals at the ages  $\bar{a}_1, \dots, \bar{a}_N$ . Typically, each value function  $s \rightarrow P_i(t, s)$  can be chosen linear in its second argument, but the present framework applies also to the more general polynomial case. The former term in the right hand side of (1.2), namely  $\int_0^{\bar{a}} P(a, J(T, a)) da$ , accounts for the total amount of the  $J$  population at time  $T$  and it can also be seen as the capital consisting of the biological resource at time  $T$ . Neglecting this term obviously leads to optimal strategies that leave no juveniles at the final time  $T$ . The value function  $j \rightarrow P(t, j)$  is also assumed to be polynomial, see Section 3.3.

To model the various costs, we use a general integral functional of the form

$$\begin{aligned} \mathcal{C}(\eta, \vartheta; T) &= \int_0^T \int_0^{\bar{a}} C_J(t, a, J(t, a)) da dt + \int_0^T \int_{\bar{a}}^{+\infty} C_S(t, a, S(t, a)) da dt \\ &\quad + \int_0^T \int_{\bar{a}}^{+\infty} C_R(t, a, R(t, a)) da dt. \end{aligned} \quad (1.3)$$

The cost functions  $w \rightarrow C_u(t, a, w)$ , for  $u \in \{J, S, R\}$ , are assumed to be polynomial in  $w$ , for all  $a$  and  $t$ . In the simplest case of *linear* cost and income, (1.2) and (1.3) reduce to

$$\mathcal{I}(\eta, \vartheta; T) = \int_0^{\bar{a}} p(a) J(T, a) da + \sum_{i=1}^N \int_0^T p_i(t) (1 - \vartheta_i(t)) S(t, \bar{a}_i -) dt. \quad (1.4)$$

$$\begin{aligned} \mathcal{C}(\eta, \vartheta; T) &= \int_0^T \int_0^{\bar{a}} c_J(t, a) J(t, a) da dt + \int_0^T \int_{\bar{a}}^{+\infty} c_S(t, a) S(t, a) da dt \\ &\quad + \int_0^T \int_{\bar{a}}^{+\infty} c_R(t, a) R(t, a) da dt. \end{aligned} \quad (1.5)$$

Here,  $p(a)$  is the unit value of juveniles of age  $a$ , while  $p_i(t)$  is the price at time  $t$  per each individual of the population  $S$  sold at maturity  $\bar{a}_i$ . Similarly, the quantity  $c_u(t, a)$ , for  $u \in \{J, S, R\}$ , is the unit cost related to the keeping of individuals of the population  $u$ , of age  $a$ , at time  $t$ .

Below, we provide the essential tools to establish effective numerical procedures able to actually compute the profit

$$\mathcal{P}(\eta, \vartheta; T) = \mathcal{I}(\eta, \vartheta; T) - \mathcal{C}(\eta, \vartheta; T). \quad (1.6)$$

as a function of the (open loop) control parameters  $\eta$  and  $\vartheta$ . In particular, this also allows to find choices of the time dependent control parameters  $\eta$  and  $\vartheta$  that allow to maximize  $\mathcal{P}$ . Moreover, the procedures presented below provide an alternative to the use of *bang-bang* controls. For a comparison between the two techniques we refer to Section 3.3.

The next section presents the main results of this note, while specific examples are deferred to paragraphs 3.1, 3.2 and 3.3. All analytic proofs are in Section 4.

## 2 Main Results

Throughout we denote  $\mathbb{R}^+ = [0, +\infty[$ , while  $\chi_A$  is the usual characteristic function of the set  $A$ , so that  $\chi_A(x) = 1$  if and only if  $x \in A$ , whereas  $\chi_A$  vanishes outside  $A$ . The positive integers  $\kappa, m$  and  $N$  are fixed throughout, as also the positive strictly increasing real numbers  $\bar{a}, \bar{a}_1, \dots, \bar{a}_N$ . It is also of use to introduce the real intervals  $I_J = [0, \bar{a}]$ ,  $I_S = I_R = [\bar{a}, +\infty[$ , and  $I_T = [0, T]$ .

Below, for a real valued function  $u$  defined on an interval  $I$ , we call  $\text{TV}(u)$  its total variation, while  $\mathbf{BV}(I; \mathbb{R})$  is the set of real valued functions with finite total variation, namely:

$$\text{TV}(u) = \sup \left\{ \sum_{i=1}^N |u(t_i) - u(t_{i-1})| : N \in \mathbb{N}, t_1, \dots, t_N \in I \text{ and } t_{i-1} < t_i \text{ for all } i \right\}$$

$$\mathbf{BV}(I; \mathbb{R}) = \{u: I \rightarrow \mathbb{R} : \text{TV}(u) < +\infty\} \text{ and } \mathbf{BV}(I; \mathbb{R}^+) = \{u: I \rightarrow \mathbb{R}^+ : \text{TV}(u) < +\infty\}.$$

We posit the following assumptions:

**(A)** For  $u = J, S, R$ , the growth rate  $g_u$  and mortality rate  $d_u$  satisfy

$$\begin{aligned} g_u &\in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(I_T \times I_u; [\check{g}_u, +\infty[) & \text{and} & \sup_{t \in \mathbb{R}^+} \text{TV}(g_u(t, \cdot)) < +\infty, \\ d_u &\in (\mathbf{C}^1 \cap \mathbf{L}^\infty)(I_T \times I_u; \mathbb{R}) & \text{and} & \sup_{t \in \mathbb{R}^+} \text{TV}(d_u(t, \cdot)) < +\infty, \end{aligned}$$

for a suitable  $\check{g}_u > 0$ , while the fertility function  $w$  satisfies  $w \in \mathbf{C}_c^1([\bar{a}, +\infty[; \mathbb{R}^+)$ .

**(ID)**  $J_o \in \mathbf{BV}(I_J; \mathbb{R}^+)$ ,  $S_o \in (\mathbf{L}^1 \cap \mathbf{BV})(I_S; \mathbb{R}^+)$  and  $R_o \in (\mathbf{L}^1 \cap \mathbf{BV})(I_R; \mathbb{R}^+)$ .

**(P)**  $P \in \mathbf{L}_{\text{loc}}^\infty([0, \bar{a}] \times \mathbb{R}^+; \mathbb{R})$  and  $P_i \in \mathbf{L}_{\text{loc}}^\infty(I_T \times \mathbb{R}^+; \mathbb{R})$  for  $i = 1, \dots, N$ . Moreover, the map  $j \rightarrow P(a, j)$ , respectively  $s \rightarrow P_i(t, s)$  for  $i = 1, \dots, N$ , is a polynomial of degree at most  $\kappa$  in  $j$  for all  $a \in [0, \bar{a}]$ , respectively in  $s$  for  $t \in I_T$ .

**(C)**  $C_u \in \mathbf{L}_{\text{loc}}^1(I_T \times I_u \times \mathbb{R}; \mathbb{R})$  and the map  $v \rightarrow C_u(t, a, v)$  is a polynomial of degree at most  $\kappa$  in  $v$ , for  $u = J, S, R$ .

Above, the restriction to  $\mathbb{R}^+$  of the initial data is not necessary from the analytic point of view, but it is justified by the biological meaning of the variables. Clearly, the extension to the case of polynomials with different degrees is essentially a mere problem of notation.

Recall, as in [6, 11], the strictly increasing sequence of *generation times*  $T_\ell$  recursively defined for  $\ell \in \mathbb{N}$ , by

$$T_0 = 0 \quad \text{and} \quad \mathcal{A}_J(T_\ell; T_{\ell-1}, 0) = \bar{a} \quad \text{or, equivalently,} \quad \mathcal{T}_J(\bar{a}; T_{\ell-1}, 0) = T_\ell, \quad (2.1)$$

the characteristic functions  $\mathcal{A}_J$  and  $\mathcal{T}_J$  being defined in (4.3) for  $u = J$ . If  $g_J$  satisfies **(A)**, then the sequence  $T_\ell$  is well defined and  $T_\ell \rightarrow +\infty$  as  $\ell \rightarrow +\infty$ . The interval  $[T_{\ell-1}, T_\ell]$  is the time period when the juveniles of the  $\ell$ -th generation are born.

The following results apply to the case of a constant  $\eta$  and a constant  $\vartheta$ , when system (1.1) fits into [5, Theorem 2.4] and turns out to be well posed in  $\mathbf{L}^1$ .

**Lemma 2.1** ([5, Corollary 3.4]). *Let  $(\mathbf{A})$  hold. For every  $\eta \in [0, 1]$ ,  $\vartheta \in [0, 1]^N$  and every initial data  $(J_o, S_o, R_o)$  as in  $(\mathbf{ID})$ , system (1.1) admits a unique solution  $(J, S, R)$  such that*

$$\begin{aligned} J(t, a) &\geq 0, & t \in I_T, & a \in I_J, \\ S(t, a) &\geq 0, & t \in I_T, & a \in I_S, \\ R(t, a) &\geq 0, & t \in I_T, & a \in I_R, \end{aligned}$$

and the stability estimates in [5, Theorem 2.4 and Theorem 2.5] hold.

In order to exhibit the existence and to actually find a value of  $\eta$  and  $\vartheta$  that maximizes  $\mathcal{P}$  as defined in (1.6), we first investigate the regularity of  $\mathcal{I}$  and  $\mathcal{C}$ , defined in (1.2) and (1.3), as functions of the control parameters  $\eta$  and  $\vartheta$ .

**Lemma 2.2** ([11, Theorem 2.2]). *Let  $(\mathbf{A})$  hold. Let  $C_J, C_S, C_R$  satisfy  $(\mathbf{C})$  and the functions  $P$  and  $P_i$  satisfy  $(\mathbf{P})$ . For every  $T > 0$ , every  $\eta \in [0, 1]$ , every  $\vartheta \in [0, 1]^N$  and every initial data  $(J_o, S_o, R_o)$  as in  $(\mathbf{ID})$ ,*

1. *the maps  $\eta \rightarrow J(T, \cdot)$ ,  $\eta \rightarrow S(T, \cdot)$ ,  $\eta \rightarrow R(T, \cdot)$ , and  $\eta \rightarrow \mathcal{I}(\eta, \vartheta; T)$  are all polynomials in  $\eta$ ;*
2. *the maps  $\vartheta \rightarrow J(T, \cdot)$ ,  $\vartheta \rightarrow S(T, \cdot)$ ,  $\vartheta \rightarrow R(T, \cdot)$  are affine in each component  $\vartheta_i$  of  $\vartheta$ , separately, while the map  $\vartheta \rightarrow \mathcal{I}(\eta, \vartheta; T)$  is polynomial in each component  $\vartheta_i$  of  $\vartheta$ .*

Hence, all the maps  $(\eta, \vartheta) \rightarrow \mathcal{C}(\eta, \vartheta; T)$ ,  $(\eta, \vartheta) \rightarrow \mathcal{I}(\eta, \vartheta; T)$ , and  $(\eta, \vartheta) \rightarrow \mathcal{P}(\eta, \vartheta; T)$  are continuously differentiable in both  $\eta$  and  $\vartheta$ .

When the control parameters are time dependent, the well posedness of (1.1) follows from [6, Theorem 2.1], which we recall here for completeness.

**Theorem 2.3** ([6, Theorem 2.1]). *Pose conditions  $(\mathbf{A})$ ,  $(\mathbf{ID})$ . For any  $\eta \in \mathbf{BV}(I_T; [0, 1])$  and  $\vartheta \in \mathbf{BV}(I_T; [0, 1]^N)$ , system (1.1) admits a unique solution. Moreover,*

$$\begin{aligned} J(t, a) &\geq 0, & t \in I_T, & a \in I_J, \\ S(t, a) &\geq 0, & t \in I_T, & a \in I_S, \\ R(t, a) &\geq 0, & t \in I_T, & a \in I_R, \end{aligned}$$

and there exists a function  $\mathcal{K} \in \mathbf{C}^0(I_T; \mathbb{R}^+)$ , with  $\mathcal{K}(0) = 0$ , dependent only on  $g_J, g_S, g_R, d_J, d_S, d_R$  and  $w$  such that for all initial data  $(J'_o, S'_o, R'_o)$  and  $(J''_o, S''_o, R''_o)$  and for all controls  $\eta', \eta'', \vartheta'$  and  $\vartheta''$ , the corresponding solutions  $(J', S', R')$  and  $(J'', S'', R'')$  to (1.1) satisfy, for every  $t \in I_T$ , the following stability estimate:

$$\begin{aligned} &\|J'(t) - J''(t)\|_{\mathbf{L}^1(I_J; \mathbb{R})} + \|S'(t) - S''(t)\|_{\mathbf{L}^1(I_S; \mathbb{R})} + \|R'(t) - R''(t)\|_{\mathbf{L}^1(I_R; \mathbb{R})} \\ &\leq \mathcal{K}(t) \left( \|J'_o - J''_o\|_{\mathbf{L}^1(I_J; \mathbb{R})} + \|S'_o - S''_o\|_{\mathbf{L}^1(I_S; \mathbb{R})} + \|R'_o - R''_o\|_{\mathbf{L}^1(I_R; \mathbb{R})} \right) \\ &\quad + t \mathcal{K}(t) \left( \|J'_o - J''_o\|_{\mathbf{L}^\infty(I_J; \mathbb{R})} + \|S'_o - S''_o\|_{\mathbf{L}^\infty(I_S; \mathbb{R})} + \|R'_o - R''_o\|_{\mathbf{L}^\infty(I_R; \mathbb{R})} \right) \\ &\quad + \mathcal{K}(t) \left( \|\eta' - \eta''\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} + \|\vartheta' - \vartheta''\|_{\mathbf{L}^\infty([0, t]; \mathbb{R}^N)} \right). \end{aligned}$$

Recall the following definition, which allows us to describe the form of the cost, income, and profit as functions of the controls.

**Definition 2.4** ([10, Definition 4.1.2]). A map  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is *multiaffine* if  $f(\eta)$  is affine as a function of each  $\eta_l$ , for  $l = 1, \dots, n$ , (keeping all other  $\eta_k$  fixed).

The elementary property below of multiaffine functions plays a key role in selecting those situations where a bang–bang control may yield the optimal profit. Its proof is deferred to Section 4.

**Lemma 2.5.** Let  $n \in \mathbb{N}$  and  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be multiaffine and not constant. Then,  $f$  admits neither points of strict local minimum, nor points of strict local maximum. Hence,  $\max_{[0,1]^n} f$  is attained on a vertex of  $[0,1]^n$ .

The two theorems below constitute the main results of the present work.

**Theorem 2.6.** Pose conditions **(A)**, **(ID)**. Introduce times  $\tau_0, \tau_1, \dots, \tau_m$  such that

$$\tau_0 = 0, \quad \tau_{k-1} < \tau_k \text{ for } k = 1, \dots, m, \quad T_\ell \notin ]\tau_{k-1}, \tau_k[ \text{ for } \begin{matrix} k = 1, \dots, m, \\ \ell \in \mathbb{N} \end{matrix} \quad (2.2)$$

and control parameters  $\eta_k \in [0, 1]$  for  $k = 1, \dots, m$ . Let  $(J, S, R)$  be the solution to (1.1) corresponding to the control

$$\eta(t) = \sum_{k=1}^m \eta_k \chi_{[\tau_{k-1}, \tau_k[}(t). \quad (2.3)$$

Then, for all  $t$  and  $a$ , the quantities  $J(t, a)$ ,  $R(t, a)$  and  $S(t, a)$  are multiaffine in  $(\eta_1, \dots, \eta_m)$ .

Remark that the latter condition  $T_\ell \notin ]\tau_{k-1}, \tau_k[$  in (2.2) can always be met, through a suitable splitting of the intervals  $[\tau_{k-1}, \tau_k]$ .

**Theorem 2.7.** Pose conditions **(A)**, **(ID)**. Introduce times  $\tau_0, \tau_1, \dots, \tau_m$  such that

$$\tau_0 = 0, \quad \tau_{k-1} < \tau_k \quad \text{for } k = 1, \dots, m$$

and control parameters  $\vartheta_i^k \in [0, 1]$  for  $k = 1, \dots, m$  and  $i = 1, \dots, N - 1$ . Let  $(J, S, R)$  be the solution to (1.1) corresponding to the controls

$$\vartheta_i(t) = \sum_{k=1}^m \vartheta_i^k \chi_{[\tau_{k-1}, \tau_k[}(t) \quad \text{for } i = 1, \dots, N - 1. \quad (2.4)$$

Then, for all  $i = 1, \dots, N - 1$ , if  $a \in ]\bar{a}_i, \bar{a}_{i+1}[$ , the quantity  $S(t, a)$  is multiaffine in the variables  $(\vartheta_1^1, \dots, \vartheta_1^m, \dots, \vartheta_i^1, \dots, \vartheta_i^m)$ .

**Corollary 2.8.** Pose conditions **(A)**, with  $g_J$  constant in time, **(ID)**, **(P)** and **(C)**. Choose controls  $\eta$  as in (2.2)–(2.3) and  $\vartheta$  as in (2.4). Then, the net profit  $\mathcal{P}$  defined in (1.6) is polynomial in  $\eta$  and  $\vartheta$  of degree at most  $\kappa$  in each of the (scalar) variables  $\eta_1, \dots, \eta_m, \vartheta_1^k, \dots, \vartheta_{N-1}^k$  separately. Moreover, globally, it is a polynomial of degree at most  $\kappa m$  in  $\eta_1, \dots, \eta_m$  and of degree at most  $\kappa m (N - 1)$  in  $\vartheta_1^k, \dots, \vartheta_{N-1}^k$ .

Thanks to the form of the costs and of the gains ensured by **(P)** and **(C)**, the proof is an immediate consequence of Theorem 2.6 and Theorem 2.7.

**Remark 2.9.** A direct consequence of Corollary 2.8 in the case (1.4)–(1.5) of linear gains and costs, thanks to Lemma 2.5, is that optimal controls  $\vartheta(t) = (\vartheta_1, \dots, \vartheta_{N-1})(t)$ , among those of the form (2.4), can be found restricting the search to only bang–bang controls, i.e., to those assuming only the values 0 and 1. Nevertheless, in [6, Theorem 1.8], it is proved that bang–bang controls well approximate the optimal ones, found in the class of  $\mathbf{BV}(I_T; [0, 1])$  for  $\eta$  and of  $\mathbf{BV}(I_T; [0, 1]^N)$  for  $\vartheta$ , provided the cost and income are linear, i.e. in the form (1.4)–(1.5).

### 3 Examples

The examples in paragraphs 3.1 and 3.2 rely on several numerical integrations of (1.1). They were accomplished using the explicit formula (4.2). To compute the gains and the costs (1.2)–(1.3), we used the standard trapezoidal rule.

For simplicity, we assume throughout that at age  $\bar{a}_N$  all the population  $S(t, \bar{a}_N)$  is sold; this corresponds to the case  $\vartheta_N \equiv 0$ .

#### 3.1 A Generational Control

We particularize Theorem 2.3 to the case of  $\eta$  as in (2.2)–(2.3) with  $\tau_\ell = T_\ell$ , so that  $\eta$  is constant on each generation. On the other hand, we keep  $\vartheta$  constant.

**Corollary 3.1.** *Pose conditions (A), (ID), (P) and (C). Choose linear gains and costs as in (1.4)–(1.5). Let  $T_0, T_1, \dots, T_n$  be as in (2.1). Set*

$$\eta(t) = \sum_{\ell=1}^n \eta_\ell \chi_{[T_{\ell-1}, T_\ell]}(t) \quad (3.1)$$

*and let  $\vartheta$  be constant. Then, the net profit  $\mathcal{P}$  defined in (1.6) is multiaffine in  $(\eta_1, \dots, \eta_n)$ . Therefore, the optimal profit can be obtained through a bang–bang control.*

In the present case (3.1) there are  $2^n$  distinct bang–bang controls: Corollary 3.1 ensures that one of them yields the maximum profit. At the same time, the profit  $\mathcal{P}$  is a multiaffine function in  $\eta$ , so that it contains at most  $2^n$  terms. Therefore, the integration of  $2^n$  suitable instances of (1.1) permits to obtain all the coefficients in the expression of  $\mathcal{P}$  as a function of  $\eta$  and, hence, to compute  $\mathcal{P}$  for *all* (i.e., not necessarily bang–bang) possible controls (3.1).

Consider the situation  $n = 2$  corresponding to the time interval  $[0, T_2]$ , we have

$$\eta(t) = \eta_1 \chi_{[0, T_1]}(t) + \eta_2 \chi_{[T_1, T_2]}(t)$$

and Corollary 3.1 ensures that the profit  $\mathcal{P}$  defined at (1.4)–(1.5)–(1.6) is actually a multiaffine function of  $\eta \equiv (\eta_1, \eta_2)$ , so that

$$\begin{aligned} \mathcal{P}(\eta_1, \eta_2) &= \mathcal{P}(0, 0) + (\mathcal{P}(1, 0) - \mathcal{P}(0, 0)) \eta_1 + (\mathcal{P}(0, 1) - \mathcal{P}(0, 0)) \eta_2 \\ &\quad + (\mathcal{P}(1, 1) - \mathcal{P}(1, 0) - \mathcal{P}(0, 1) + \mathcal{P}(0, 0)) \eta_1 \eta_2. \end{aligned}$$

In other words, thanks to the qualitative information provided by Corollary 3.1, computing  $\mathcal{P}$  only 4 times allows to obtain the expression of  $\mathcal{P}(\eta_1, \eta_2)$  valid for all  $(\eta_1, \eta_2) \in [0, 1]^2$ .

As an example, we consider the setting (1.1)–(1.4)–(1.5) defined by the choices:

$$\begin{aligned}
g_J(t, a) &= 1.00 & d_J(t, a) &= 1.50 & c_J(t, a) &= 0.25 & J_o(a) &= 1.00 \\
g_S(t, a) &= 1.00 & d_S(t, a) &= 0.50 & c_S(t, a) &= 0.00 & S_o(a) &= 0.00 \\
g_R(t, a) &= 1.00 & d_R(t, a) &= 0.75 & c_R(t, a) &= 0.00 & R_o(a) &= 0.00 \\
\bar{a} &= 1.00 & \bar{a}_1 &= 1.50 & N &= 1 \\
p(a) &= 0.00 & p_1(t) &= 8.00 & w(a) &= 120.00 \chi_{[1.00, 4.00]}(a).
\end{aligned}$$

Using the expression (4.2) of the exact solution to (1.1) we obtain (up to the second decimal digit)

$$P(0, 0) = -19.97, \quad P(1, 0) = 3.13, \quad P(0, 1) = 8.22, \quad P(1, 1) = 3.13,$$

so that

$$\mathcal{P}(\eta_1, \eta_2) = -19.97 + 23.10 \eta_1 + 28.18 \eta_2 - 28.18 \eta_1 \eta_2. \quad (3.2)$$

Coherently with the results above, the maximum of  $\mathcal{P}$  is attained at the bang–bang control

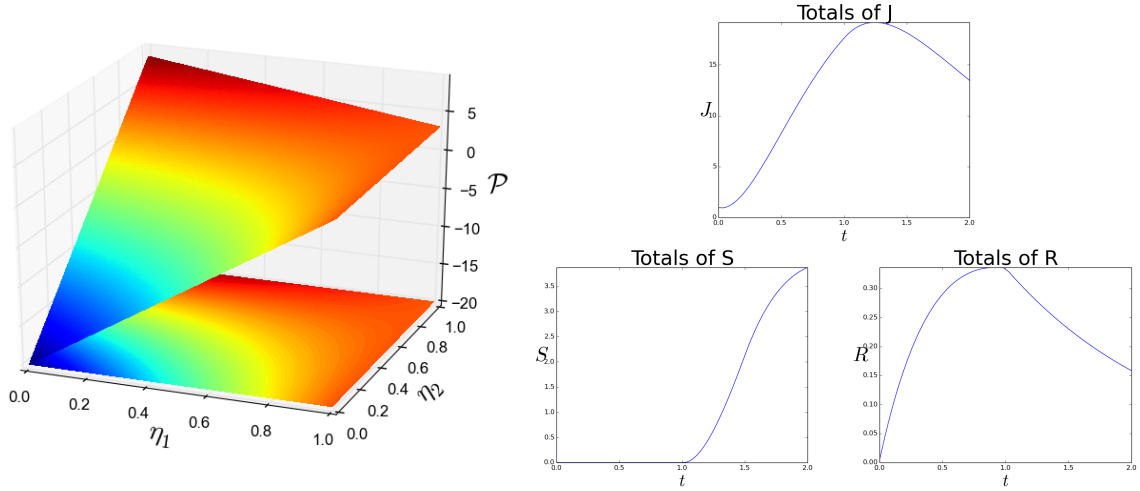


Figure 2: Left, graph of the profit (3.2): the maximum value  $\mathcal{P} = 8.21$  on  $[0, 1]^2$  is attained at  $(\eta_1, \eta_2) = (0, 1)$ . Right, the total amounts of the different populations as a function of time: top,  $J$  and, bottom,  $S$  and  $R$ .

$(\eta_1, \eta_2) = (0, 1)$ , see Figure 2. This strategy amounts to first keep all juveniles for reproduction and then sell them all.

### 3.2 A Periodic Control

We now consider the case of a growth function  $g_J$  independent of time. Then, with reference to (2.1), we have  $T_\ell = \ell T_1$  for all  $\ell \in \mathbb{N}$ . It is then natural to consider a piecewise constant control which is periodic and with period  $T_1$ :

$$\begin{aligned}
\eta(t) &= \eta(\tau) && \text{whenever } (t - \tau)/T_1 \in \mathbb{N} \\
\eta(t) &= \sum_{h=1}^m \eta_h \chi_{[\tau_{h-1}, \tau_h[}(t) && \text{if } 0 \leq \tau_{h-1} < \tau_h \leq T_1 \text{ for } h = 1, \dots, m \text{ and } t \in [0, T_1].
\end{aligned} \quad (3.3)$$



**Corollary 3.2.** *Pose conditions (A), (ID), (P) and (C). Assume that the growth function  $g_J$  is independent of time. Choose  $\eta$  as in (3.3) with  $T = T_n$  for a given  $n \in \mathbb{N} \setminus \{0\}$  and let  $\vartheta$  be constant. Then, the net profit  $\mathcal{P}$  defined in (1.4)–(1.5)–(1.6) is a polynomial of degree at most  $n$  in  $(\eta_1, \eta_2, \dots, \eta_m)$ .*

The proof is a direct consequence of Theorem 2.6 and is hence omitted. Observe that a polynomial of degree  $n$  in  $m$  variables contains at most  $\nu = \binom{n+m}{n}$  terms. Therefore, Corollary 3.2 reduces the problem of maximizing (1.6) along the solutions to (1.1) to:

1. the computation of  $\nu$  solutions to (1.1),
2. the solution to a linear system of  $\nu$  equations in  $\nu$  variables,
3. the maximization of a polynomial.

Consider the following example. In the case  $m = 2$  in (3.3), choosing the time interval  $[0, T_2]$ , i.e.,  $n = 2$ , we set

$$\eta(t) = \eta_1 \chi_{[0.0, 0.5]}(t) + \eta_2 \chi_{[0.5, 1.0]}(t) + \eta_1 \chi_{[1.0, 1.5]}(t) + \eta_2 \chi_{[1.5, 2.0]}(t), \quad (3.4)$$

corresponding to  $\tau_0 = 0.0$ ,  $\tau_1 = 0.5$  and  $\tau_2 = 1.0$ . Corollary 3.2 ensures that  $\mathcal{P}$  is a polynomial of degree at most 2 separately in  $\eta_1$  and  $\eta_2$ , so that

$$\mathcal{P}(\eta_1, \eta_2) = c_0 + c_1 \eta_1 + c_2 \eta_2 + c_3 \eta_1 \eta_2 + c_4 \eta_1^2 + c_5 \eta_2^2 \quad (3.5)$$

and  $\nu = 6$  numerical integrations of (1.1) with the consequent evaluation of (1.6) allow to obtain the coefficients  $c_0, \dots, c_5$  and, hence, the full knowledge of the profit as a function of the control parameters.

We consider now the setting (1.1)–(1.4)–(1.5) defined by the choices:

$$\begin{array}{llllll} \bar{a} = 1.00 & d_J(t, a) = 0.50 & c_J(t, a) = 0.25 & p(a) = 1.00 & J_o(a) = 1.00 \\ N = 1 & d_S(t, a) = 1.00 & c_S(t, a) = 0.25 & p_1(t) = 8.20 & S_o(a) = 0.00 \\ \bar{a}_1 = 1.50 & d_R(t, a) = 1.50 & c_R(t, a) = 0.25 & w(a) = 10.00 \chi_{[1.00, 4.00]}(a) & R_o(a) = 0.00. \end{array}$$

Using the expression of the exact solution to (1.1) we obtain (up to the second decimal digit)

$$c_0 = 3.65, \quad c_1 = 0.46, \quad c_2 = -0.88, \quad c_3 = 1.11, \quad c_4 = -1.06, \quad c_5 = 0.46. \quad (3.6)$$

The resulting profit is plotted in Figure 3 as a function of  $(\eta_1, \eta_2)$ . Remark that the resulting optimal control is *not* bang–bang. At the times  $t = 0.50, 1.00, 1.50$  the sharp changes in the graphs of  $J, S$  and  $R$  are due to the sharp changes in the control, as prescribed in (3.4).

### 3.3 A Stabilizing Strategy

As a further example, we consider the case of a nonlinear profit. A justification for this choice can be the necessity to stabilize the juvenile population to reduce the running costs caused by the  $J$  population.

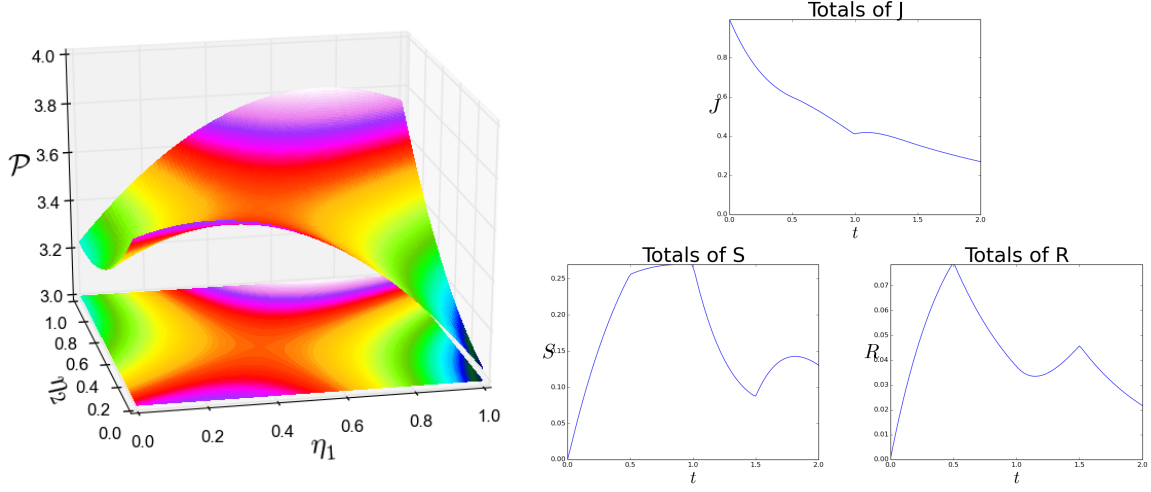


Figure 3: Left, graph of the polynomial (3.5)–(3.6): the maximum gain  $\mathcal{P} = 3.81$  on  $[0, 1]^2$  is attained at  $(\eta_1, \eta_2) = (0.74, 1.00)$ . Right, the total amounts of the different populations as a function of time: top,  $J$  and, bottom,  $S$  and  $R$ .

Therefore, we consider system (1.1), with an income function of the type (1.2) and a nonlinear cost for the  $J$  population given by

$$\mathcal{C}(\eta, \vartheta; T) = - \int_0^T \int_0^{\bar{a}} (J(t, a) - \bar{J})^2 da dt. \quad (3.7)$$

Here, the fixed parameter  $\bar{J}$  can be seen as the juvenile density that, say, minimizes the running costs. We are thus lead to the maximization of the profit (1.6), with linear income (1.4) and cost (3.7). Let  $T_\ell$  be as in (2.1) and consider a generational control  $\eta$  as in (3.1), and piecewise constant controls  $\vartheta_i$  ( $i \in \{1, \dots, N-1\}$ ) as

$$\vartheta_i(t) = \sum_{\ell=1}^n \vartheta_i^\ell \chi_{[T_{\ell-1}, T_\ell]}(t), \quad (3.8)$$

where  $\vartheta_i^\ell \in [0, 1]$  for every  $i \in \{1, \dots, N-1\}$  and  $\ell \in \{1, \dots, n\}$ . Then, by the analysis in Section 2, we can assert that the profit (1.6) is a second order polynomial in  $\eta_1, \dots, \eta_n$  whose first and zeroth order terms are multiaffine in  $\vartheta_1^\ell, \dots, \vartheta_{N-1}^\ell$ :

$$\begin{aligned} \mathcal{P}(\eta, \vartheta) &= \sum_{\lambda \in \{0,1\}^n} \sum_{\ell \in \{1, \dots, n\}^{N-1}} \sum_{\beta \in \{0,1\}^{N-1}} c_{\lambda, \ell, \beta} \eta^\lambda (\vartheta_1^{\ell_1})^{\beta_1} \dots (\vartheta_{N-1}^{\ell_{N-1}})^{\beta_{N-1}} \\ &+ \sum_{\lambda \in \{0,1,2\}^n: \max \lambda = 2} c'_\lambda \eta^\lambda \end{aligned} \quad (3.9)$$

which is a polynomial defined by

$$\nu = n^{N-1} 2^{n+N-1} + 3^n - 2^n \quad (3.10)$$

real coefficients. Thus, solving  $\nu$  times the renewal equations (1.1), computing the corresponding  $\nu$  profits (3.9), solving a  $\nu \times \nu$  linear system to get the  $\nu$  coefficients allows to obtain

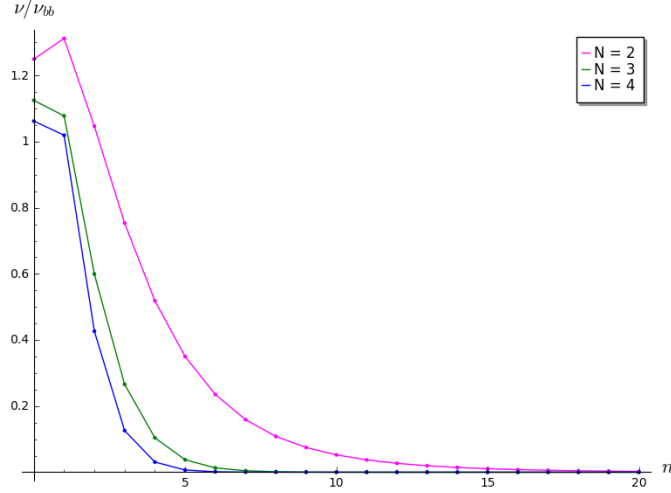


Figure 4: Ratio  $\nu/\nu_{bb}$  as a function of the number of generations  $n$ . As in (3.10),  $\nu$  is the number of integrations of (1.1) that are necessary to compute the coefficients of  $\mathcal{P}$  in (3.9) as a function of  $\eta$  as in (3.1) and  $\vartheta$  as in (3.8). Here,  $\nu_{bb} = 2^{nN}$  is the total number of bang–bang controls.

an expression for  $\mathcal{P}$  valid for *all* possible control parameters  $\eta \in [0, 1]^n$ ,  $\vartheta \in [0, 1]^{n(N-1)}$ . As a comparison, we remark that the total number of bang–bang controls in the present case is  $\nu_{bb} = 2^{nN}$  and there is no guarantee that the optimal control is of bang–bang type. For a comparison between  $\nu$  and  $\nu_{bb}$ , refer to Figure 4.

## 4 Technical Details

As in [5, 6, 12], we recall that the initial – boundary value problem for the renewal equation

$$\begin{cases} \partial_t u + \partial_a (g_u(t, a) u) = d_u(t, a) u & t \geq 0 \\ u(0, a) = u_o(a) & a \geq a_u \\ g_u(t, a_u) u(t, a_u+) = b(t) & \end{cases} \quad (4.1)$$

admits a unique solution that can be explicitly computed integrating along characteristics as

$$u(t, a) = \begin{cases} u_o(\mathcal{A}_u(0; t, a)) \psi_u(0, t, a) & a \geq \mathcal{A}_u(t; 0, a_u) \\ \frac{b(\mathcal{T}_u(a_u; t, a))}{g_u(\mathcal{T}_u(a_u; t, a), a_u)} \psi_u(\mathcal{T}_u(a_u; t, a), t, a) & a < \mathcal{A}_u(t; 0, a_u), \end{cases} \quad (4.2)$$

where the maps  $t \rightarrow \mathcal{A}_u(t, t_o, a_o)$  and  $a \rightarrow \mathcal{T}_u(a; t_o, a_o)$ , with  $t \in \mathbb{R}^+$  and  $a, a_o \in I_u$ , are defined as

$$\begin{aligned} t \rightarrow \mathcal{A}_u(t; t_o, a_o) & \text{ is the solution to } \begin{cases} \dot{a} = g_u(t, a) \\ a(t_o) = a_o \end{cases} \quad \text{and} \\ a \rightarrow \mathcal{T}_u(a; t_o, a_o) & \text{ is its inverse, i.e., } \mathcal{A}_u(\mathcal{T}_u(a; t_o, a_o); t_o, a_o) = a \quad \text{for all } a \in I_u, \end{aligned} \quad (4.3)$$

while the map  $\psi_u$  is given by

$$\psi_u(t_1, t_2, a) = \exp \int_{t_1}^{t_2} \left( d_u(s, \mathcal{A}_u(s; t_2, a)) - \partial_a g_u(s, \mathcal{A}_u(s; t_2, a)) \right) ds. \quad (4.4)$$

Clearly, the knowledge of the maps  $\mathcal{A}_u$ ,  $\mathcal{T}_u$  and  $\psi_u$  does not require knowledge of the solution to (4.1) but relies only on the solution to the ordinary differential equation (4.3).

**Proof of Lemma 2.5.** The proof is by induction on  $n$ . If  $n = 1$ , then  $f(\eta) = a + b\eta$  and the proof follows by basic calculus. Let now  $n > 1$ . Assume that  $\bar{\eta} \in \mathbb{R}^{n+1}$  is a point of strict local maximum or minimum for the multiaffine function  $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ . Then, one can write

$$f(\eta_1, \dots, \eta_{n+1}) = a(\eta_1, \dots, \eta_n) + b(\eta_1, \dots, \eta_n)(\eta_{n+1} - \bar{\eta}_{n+1})$$

for suitable multiaffine functions  $a, b: \mathbb{R}^n \rightarrow \mathbb{R}$ . Since  $a(\eta_1, \dots, \eta_n) = f(\eta_1, \dots, \eta_n, \bar{\eta}_{n+1})$  has a point of strict local maximum or minimum at  $(\bar{\eta}_1, \dots, \bar{\eta}_n)$ , by the inductive assumption the map  $a$  is constant. Since, the map  $\eta_{n+1} \rightarrow b(\bar{\eta}_1, \dots, \bar{\eta}_n)(\eta_{n+1} - \bar{\eta}_{n+1})$  may not attain a strict local maximum or minimum at  $\eta_{n+1} = \bar{\eta}_{n+1}$ , the proof is completed.  $\square$

**Proof of Theorem 2.6 and Proof of Theorem 2.7.** Fix an arbitrary time  $\tau \geq 0$ . Lengthy but elementary computations based on Figure 5 show that the  $J$  component of the solution to (1.1) admits the following representation, for  $t \in [\tau, \tau + \bar{a}]$  and where we used (4.2)–(4.3)–(4.4) for  $u = J, S, R$ :

$$J(t, a) = \begin{cases} J(\tau, \mathcal{A}_J(\tau; t, a)) \psi_J(\tau, t, a) & a \in [\mathcal{A}_J(t; \tau, 0), \bar{a}] \\ \frac{1}{g_J(\mathcal{T}_J(0; t, a))} \int_{\bar{a}}^{\mathcal{A}_R(\mathcal{T}_J(0; t, a); \tau, a)} w(\alpha) \\ \quad \times R(\tau, \mathcal{A}_R(\tau, \mathcal{T}_J(0; t, a); \tau, \alpha)) \psi_R(\tau, \mathcal{T}_J(0; t, a); \tau, \alpha) d\alpha \\ \quad + \frac{1}{g_J(\mathcal{T}_J(0; t, a))} \int_{\mathcal{A}_R(\mathcal{T}_J(0; t, a); \tau, a)}^{+\infty} w(\alpha) \\ \quad \times \left(1 - \eta(\mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha))\right) \frac{g_J(\mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha), \bar{a})}{g_R(\mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha), \bar{a})} \\ \quad \times J(\tau, \mathcal{A}_J(\tau; \mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha), \bar{a})) \\ \quad \times \psi_J(\tau, \mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha), \bar{a}) \\ \quad \times \psi_R(\mathcal{T}_R(\bar{a}; \mathcal{T}_J(0; t, a), \alpha), t, \alpha) d\alpha. & a \in [0, \mathcal{A}_J(t; \tau, 0)] \end{cases} \quad (4.5)$$

The  $R$  population is given by

$$R(t, a) = \begin{cases} R(\tau, \mathcal{A}_R(\tau, t, a)) \psi_R(\tau, t, a) & a \geq \mathcal{A}_R(t; \tau, \bar{a}) \\ \left(1 - \eta(\mathcal{T}_R(\bar{a}; t, a))\right) \frac{g_J(\mathcal{T}_R(\bar{a}; t, a), \bar{a})}{g_R(\mathcal{T}_R(\bar{a}; t, a), \bar{a})} \\ \quad \times J(\tau, \mathcal{A}_J(\tau; \mathcal{T}_R(\bar{a}; t, a), \bar{a})) \\ \quad \times \psi_J(\tau, \mathcal{T}_R(\bar{a}; t, a), \bar{a}) \psi_R(\mathcal{T}_R(\bar{a}; t, a), t, a) & a \in [\bar{a}, \mathcal{A}_R(t; \tau, \bar{a})] \end{cases} \quad (4.6)$$

and, finally, the  $S$  population for  $a \in [\bar{a}, \bar{a}_1]$  is

$$S(t, a) = \begin{cases} S(\tau, \mathcal{A}_S(\tau, t, a)) \psi_S(\tau, t, a) & a \geq \mathcal{A}_S(t; \tau, \bar{a}) \\ \eta(\mathcal{T}_S(\bar{a}; t, a);) \frac{g_J(\mathcal{T}_S(\bar{a}; t, a), \bar{a})}{g_S(\mathcal{T}_S(\bar{a}; t, a), \bar{a})} \\ \quad \times J(\tau, \mathcal{A}_J(\tau; \mathcal{T}_S(\bar{a}; t, a), \bar{a})) \\ \quad \times \psi_J(\tau, \mathcal{T}_S(\bar{a}; t, a), \bar{a}) \psi_S(\mathcal{T}_S(\bar{a}; t, a), t, a). & a \in [\bar{a}, \mathcal{A}_S(t; \tau, \bar{a})] \end{cases} \quad (4.7)$$

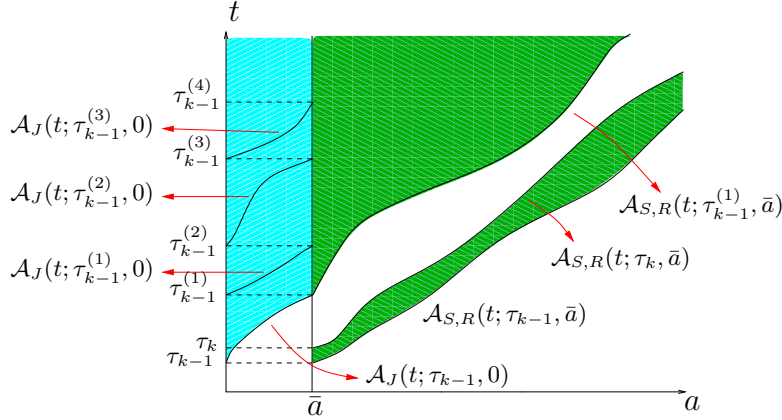


Figure 5: In the white regions, the quantities  $(J, S, R)(t, a)$  are independent of  $\eta_k$ . For  $a \in [0, \bar{a}]$ , in the shaded region  $J(t, a)$  is at most first order in  $\eta_k$ . Similarly, for  $a > \bar{a}$ , the shaded region describes where  $S(t, a)$  or  $R(t, a)$  may depend on  $\eta_k$ , at most at the first order.

The expression of  $S$  for  $a \geq \bar{a}_1$  directly follows. Note that the right hand side in the explicit expression above depends only on the values attained by  $(J, S, R)$  at  $t = \tau$ .

Fix now an index  $k$ . Clearly,  $J(t, a)$ ,  $S(t, a)$  and  $R(t, a)$  are all independent of  $\eta_k$  for  $t \in [0, \tau_{k-1}]$ . Consider the time interval  $[\tau_{k-1}, \mathcal{T}_J(\bar{a}; \tau_{k-1}, 0)]$ . By (4.5), see also Figure 5, it is clear that  $J(t, a)$  is independent of  $\eta_k$  for

$$(t, a) \in \{(\tau, \alpha) : \tau \in [\tau_{k-1}, \mathcal{T}_J(\bar{a}; \tau_{k-1}, 0)] \text{ and } \alpha \geq \mathcal{A}_J(\tau; \tau_{k-1}, 0)\}.$$

Clearly,  $S(t, a)$ , respectively  $R(t, a)$ , is independent of  $\eta_k$  whenever  $a \geq \mathcal{A}_S(t; \tau_{k-1}, \bar{a})$ , respectively  $a \geq \mathcal{A}_R(t; \tau_{k-1}, \bar{a})$ .

On the strip  $\{(t, a) : t \in [\mathcal{T}_S(a; \tau_{k-1}, \bar{a}), \mathcal{T}_S(t; \tau_k, \bar{a})] \text{ and } a \geq \bar{a}\}$ , the quantity  $S(t, a)$  is linear in  $\eta_k$  by (4.7). Similarly, on  $\{(t, a) : t \in [\mathcal{T}_R(a; \tau_{k-1}, \bar{a}), \mathcal{T}_R(t; \tau_k, \bar{a})] \text{ and } a \geq \bar{a}\}$ , by (4.6)  $R(t, a)$  is linear in  $(1 - \eta_k)$ . Again by (4.6) and (4.7),  $S(t, a)$ , respectively  $R(t, a)$ , is independent of  $\eta_k$  for  $t \in [\mathcal{T}_S(a; \tau_k, \bar{a}), \mathcal{T}_S(a; \mathcal{T}_J(\bar{a}; \tau_{k-1}, 0))]$  and  $a \geq \bar{a}$ , respectively  $t \in [\mathcal{T}_R(a; \tau_k, \bar{a}), \mathcal{T}_R(a; \mathcal{T}_J(\bar{a}; \tau_{k-1}, 0))]$  and  $a \geq \bar{a}$ . Finally, the above considerations and (4.5) ensure that  $J(t, a)$  is affine in  $\eta_k$  for  $t \in [\mathcal{T}_J(a; \tau_{k-1}, 0), \mathcal{T}_J(a; \tau_k, 0)]$  and  $a \in [0, \bar{a}]$ . The proof is thus completed for  $t \in [\tau_{k-1}, \mathcal{T}_J(\bar{a}; \tau_{k-1}, 0)]$ .

On the basis of (4.5)–(4.6)–(4.7), a straightforward iterative procedure allows to complete the proof related to the dependence of  $(J, S, R)(t, a)$  on  $\eta_k$ .

The proof concerning the dependence of  $S(t, a)$  on  $\vartheta_i^k$  directly follows from (1.1).  $\square$

**Proof of Corollary 3.1.** Apply Corollary 2.8 with  $T = T_\ell$ , use the assumption (3.1) and Lemma 2.5 to complete the proof.  $\square$

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